Topic 7 –

Second order linear homogeneous constant coefficient ODEs

Topic 7 – 2nd order linear homogeneous constant coefficient

We will now learn methods to find the solutions to 2nd order equations. We start with the simplest ones we can. These are

$$a_2 y'' + a_1 y' + a_0 y = 0$$

Where a_2, a_1, a_0 are constants, $a_2 \neq 0$

Ex:

$$y'' - 7y' + 10y = 0$$

$a_2 = 1$ $a_1 = -7$ $a_0 = 10$

Def: The characteristic equation of

$$a_2 y'' + a_1 y' + a_0 y = 0$$

is

$$a_2 r^2 + a_1 r + a_0 = 0$$

Ex: The characteristic equation of

$$y'' - 7y' + 10y = 0$$

is

$$r^2 - 7r + 10 = 0$$

Why do we do this? The roots of the characteristic equation tell us the solution to the differential equation.

Below we give formulas for how to solve $a_2 y'' + a_1 y' + a_0 y = 0$. At the end of these notes are the proofs of why these formulas work.

Formula time Consider

$$a_2 y'' + a_1 y' + a_0 y = 0 \qquad (*)$$

where a_2, a_1, a_0 are constants and $a_2 \neq 0$. There are three cases depending on the roots of the characteristic equation $a_2 r^2 + a_1 r + a_0 = 0$.

$$\boxed{\begin{array}{l}\underline{\text{case 1:}} \text{ If the characteristic equation has two distinct real roots } r_1, r_2 \text{ then the solution to } (*) \text{ is} \\ \qquad y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}\end{array}}$$

<u>case 2:</u> If the characteristic equation has a repeated real root r, then the solution to (*) is

$$y_h = c_1 e^{rx} + c_2 x e^{rx}$$

<u>case 3:</u> If the characteristic equation has imaginary roots $\alpha \pm i\beta$

α = alpha
β = beta
$i = \sqrt{-1}$

then the solution to (*) is

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

$\boxed{\begin{array}{l}\text{Ex: Solve} \\ y'' - 7y' + 10y = 0\end{array}}$

characteristic equation:

$$r^2 - 7r + 10 = 0$$

The roots are:

$$r = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2}$$

$$= \frac{7+3}{2}, \frac{7-3}{2} =$$

$$= 5, 2$$

case 1

two distinct real roots

$r_1 = 5, r_2 = 2$

Answer:

$$y_h = c_1 e^{5x} + c_2 e^{2x}$$

Ex: Solve

$$y'' - 4y' + 4y = 0$$

The characteristic equation is

$$r^2 - 4r + 4 = 0$$

The roots are:

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{0}}{2}$$

$$= \frac{4}{2} = 2$$

case 2
repeated
real
root
$r = 2$

Answer:

$$y_h = \underbrace{c_1 e^{2x} + c_2 x e^{2x}}_{c_1 e^{rx} + c_2 x e^{rx}}$$

$\boxed{\begin{array}{l}\underline{\text{Ex:}}\ \text{Solve} \\ y'' - 4y' + 13y = 0\end{array}}$

The characteristic equation is

$$r^2 - 4r + 13 = 0$$

The roots are

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm \sqrt{36}\overbrace{\sqrt{-1}}^{i}}{2}$$

$$= \frac{4 \pm 6i}{2} = \underbrace{2 \pm 3i}$$

$2+3i, 2-3i$

RECALL:

Case 3 formula:

roots: $\alpha \pm \beta i$

solution: $y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$

In our example, $\alpha \pm \beta i = 2 \pm 3i$

So, $\alpha = 2$, $\beta = 3$.

Summary! The general solution to

$$y'' - 4y' + 13y = 0$$

is

$$y_h = c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x)$$

Ex:

Solve:

$$4y'' - y' = 0$$
$$y(0) = -1$$
$$y'(0) = 1$$

First we solve

$$4y'' - y' = 0$$

The characteristic equation is

$$4r^2 - r = 0$$

Just factor

$$\underbrace{r}_{r=0}\underbrace{(4r-1)}_{4r-1=0} = 0$$

$r = 1/4$

So we get two distinct real roots $r_1 = 0$, $r_2 = 1/4$.

So, the general solution to $4y'' - y' = 0$ is:

case 1 from mon. theorem

$$y_h = c_1 e^{0x} + c_2 e^{\frac{1}{4}x}$$

$$= c_1 + c_2 e^{x/4}$$

$e^{0x} = e^0 = 1$

Now we make the solution also satisfy $y(0) = -1$, $y'(0) = 1$.

We have

$$y_h = c_1 + c_2 e^{x/4}$$

$$y_h' = \frac{1}{4} c_2 e^{x/4}$$

Need to solve:

$$\boxed{\begin{aligned} y_h(0) &= -1 \\ y_h'(0) &= 1 \end{aligned}} \longrightarrow \boxed{\begin{aligned} c_1 + c_2 e^{0/4} &= -1 \\ \frac{1}{4} c_2 e^{0/4} &= 1 \end{aligned}}$$

$$\boxed{e^{0/4} = e^0 = 1}$$

$$\boxed{\begin{aligned} c_1 + c_2 &= -1 \quad (1) \\ \frac{1}{4} c_2 &= 1 \quad (2) \end{aligned}}$$

(2) says $c_2 = 4$

Plug $c_2 = 4$ into (1) to get $c_1 + 4 = -1$

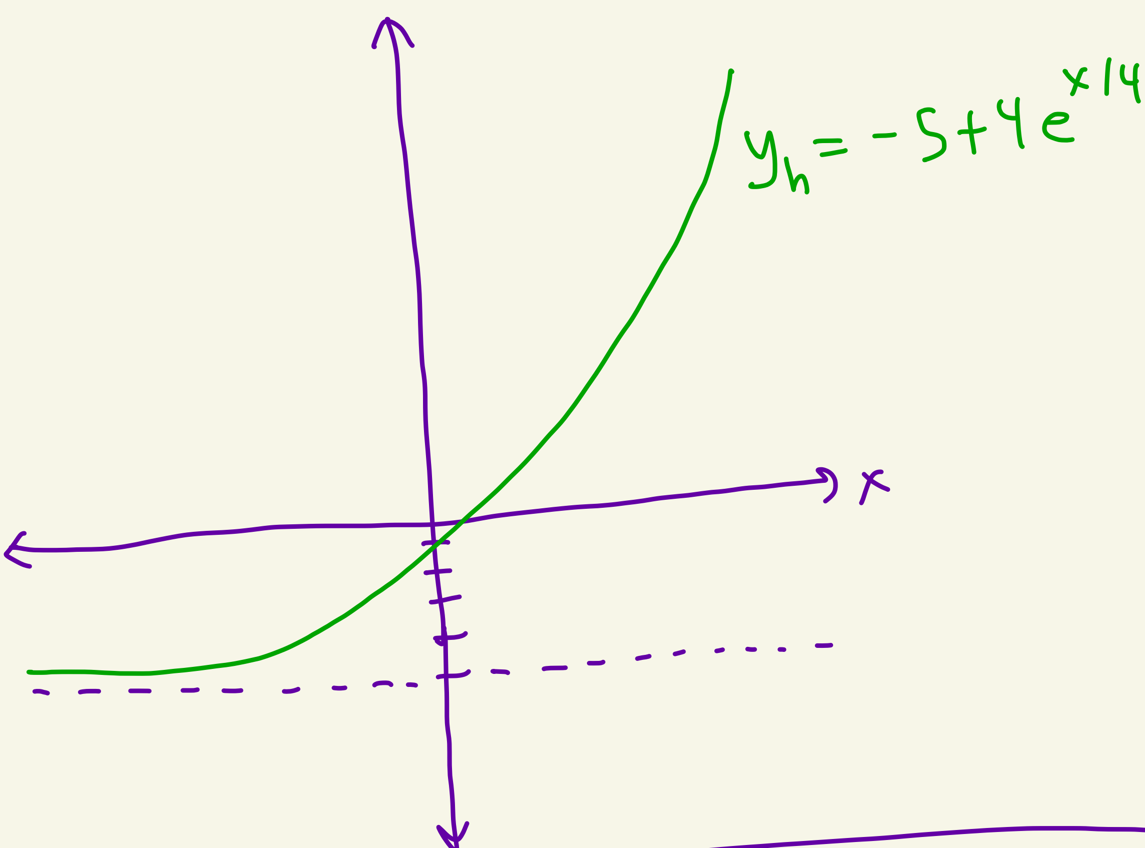
So, $c_1 = -5$.

Thus,

$$y_h = c_1 + c_2 e^{x/4}$$
$$= -5 + 4e^{x/4}$$

So, $y_h = -5 + 4e^{x/4}$ is the solution to

$$4y'' - y' = 0, \quad y(0) = -1, \; y'(0) = 1$$



Below are proofs of why the formulas given for the 3 cases above are true

Let's analyze the cases starting with cases 1 & 2.
Suppose the characteristic equation

$$a_2 r^2 + a_1 r + a_0 = 0$$

of

$$a_2 y'' + a_1 y' + a_0 y = 0$$

has a real root r.

Then,

$$a_2 r^2 + a_1 r + a_0 = 0$$

Consider the function $f(x) = e^{rx}$.

Then, $f'(x) = re^{rx}$, $f''(x) = r^2 e^{rx}$.

So, plugging f into the ODE gives

$$\begin{aligned} & a_2 f'' + a_1 f' + a_0 f \\ &= a_2 r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} \\ &= e^{rx}(a_2 r^2 + a_1 r + a_0) \\ &= e^{rx}(0) \\ &= 0 \end{aligned}$$

Thus, $f(x) = e^{rx}$ is a solution to

$$a_2 y'' + a_1 y' + a_0 y = 0$$

Case 1: Suppose r_1, r_2 are two real roots of the characteristic polynomial with $r_1 \neq r_2$. Then $f_1(x) = e^{r_1 x}$ and $f_2(x) = e^{r_2 x}$ both solve the ODE and the Wronskian is

$$W(e^{r_1 x}, e^{r_2 x}) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix}$$

$$= r_2 e^{(r_1+r_2)x} - r_1 e^{(r_1+r_2)x}$$

$$= \underbrace{(r_2 - r_1)}_{r_2 - r_1 \neq 0} \underbrace{e^{(r_1+r_2)x}}_{e^{(r_1+r_2)x} > 0} \neq 0 \text{ for any } x$$

Thus, $f_1(x) = e^{r_1 x}$ and $f_2(x) = e^{r_2 x}$ are linearly independent and every solution to $a_2 y'' + a_1 y' + a_0 y = 0$ will be of the form

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2: Suppose the characteristic polynomial of $a_2y''+a_1y'+a_0y=0$ has only one real root r_1 but it's repeated. We know from above that one solution will be $f_1(x)=e^{r_1x}$. Let's show that another solution is $f_2(x)=xe^{r_1x}$.

Since r_1 is a repeated root we get

r_1 is a repeated root

$$\begin{aligned} a_2r^2+a_1r+a_0 &= a_2(r-r_1)^2 \\ &= a_2r^2-2a_2r_1r+a_2r_1^2 \end{aligned}$$

algebra

Thus, $a_1=-2a_2r_1$ and $a_0=a_2r_1^2$.

So the ODE becomes

$$a_2y''-2a_2r_1y'+a_2r_1^2y=0$$

Let's now plug in $f_2(x)=xe^{r_1x}$.

We have

$$\begin{aligned} f_2(x) &= xe^{r_1x} \\ f_2'(x) &= e^{r_1x}+r_1xe^{r_1x} \\ f_2''(x) &= r_1e^{r_1x}+r_1e^{r_1x}+r_1^2xe^{r_1x} \end{aligned}$$

Plugging f_2 into the ODE gives

$$a_2 f_2'' - 2a_2 r_1 f_2' + a_2 r_1^2 f_2$$

$$= \cancel{a_2 r_1 e^{r_1 x}} + \cancel{a_2 r_1 e^{r_1 x}} + a_2 r_1^2 x e^{r_1 x}$$

$$- \cancel{2a_2 r_1 e^{r_1 x}} - 2a_2 r_1^2 x e^{r_1 x}$$

$$+ a_2 r_1^2 x e^{r_1 x}$$

$$= x e^{r_1 x}\left(a_2 r_1^2 - 2a_2 r_1 \cdot r_1 + a_2 r_1^2\right)$$

$$= x e^{r_1 x}\left(\underbrace{a_2 r_1^2 + a_1 r_1 + a_0}_{0}\right)$$

$$= 0$$

Thus, $f_2(x) = xe^{r_1 x}$ also solves the ODE.

The Wronskian of $f_1(x) = e^{r_1 x}$ and $f_2(x) = xe^{r_1 x}$ is

$$W(e^{r_1 x}, xe^{r_1 x}) = \begin{vmatrix} e^{r_1 x} & xe^{r_1 x} \\ r_1 e^{r_1 x} & e^{r_1 x} + r_1 x e^{r_1 x} \end{vmatrix}$$

$$= e^{2r_1 x} + r_1 x e^{2r_1 x} - r_1 x e^{2r_1 x}$$

$$= e^{2r_1 x} \neq 0 \quad \text{for any } x$$

Thus, $f_1(x) = e^{r_1 x}$ and $f_2(x) = x e^{r_1 x}$ are two linearly independent solutions to $a_2 y'' + a_1 y' + a_0 y = 0$ in this case and every solution must be of the form

$$y_h = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

Now let's work on case 3 where we have two complex roots

Case 3: Suppose the characteristic polynomial of $a_2y''+a_1y'+a_0y=0$ has two complex roots. We can divide by a_2 and we get the same equation $y''+\frac{a_1}{a_2}y'+\frac{a_0}{a_2}y=0$. For ease of derivation lets assume our equation has the form $y''+by'+cy=0$. And suppose we have two complex roots: $\alpha+i\beta$ and $\alpha-i\beta$.

We claim that $f_1(x)=e^{\alpha x}\cos(\beta x)$ and $f_2(x)=e^{\alpha x}\sin(\beta x)$ will be linearly independent solutions to the ODE.

Since $\alpha+i\beta$ and $\alpha-i\beta$ are roots we know the characteristic equation factors as follows:

$$r^2+br+c=(r-(\alpha+i\beta))(r-(\alpha-i\beta))$$
$$=r^2-2\alpha r+\alpha^2+\beta^2$$

Thus, $b=-2\alpha$ and $c=\alpha^2+\beta^2$.

Let's show $f_1(x)=e^{\alpha x}\cos(\beta x)$ solves the ODE.

We have

$$f_1(x)=e^{\alpha x}\cos(\beta x)$$

$$f_1'(x) = \alpha e^{\alpha x}\cos(\beta x) - \beta e^{\alpha x}\sin(\beta x)$$

$$\begin{aligned} f_1''(x) &= \alpha^2 e^{\alpha x}\cos(\beta x) - \alpha\beta e^{\alpha x}\sin(\beta x) \\ &\quad - \beta\alpha e^{\alpha x}\sin(\beta x) - \beta^2 e^{\alpha x}\cos(\beta x) \\ &= \alpha^2 e^{\alpha x}\cos(\beta x) - 2\alpha\beta e^{\alpha x}\sin(\beta x) \\ &\quad - \beta^2 e^{\alpha x}\cos(\beta x) \end{aligned}$$

Plugging these into the ODE gives

$$\begin{aligned} f_1'' + bf_1' + cf_1 &= f_1'' - 2\alpha f_1' + (\alpha^2+\beta^2) f_1 \\ &= \alpha^2 e^{\alpha x}\cos(\beta x) - 2\alpha\beta e^{\alpha x}\sin(\beta x) - \beta^2 e^{\alpha x}\cos(\beta x) \\ &\quad - 2\alpha^2 e^{\alpha x}\cos(\beta x) + 2\alpha\beta e^{\alpha x}\sin(\beta x) \\ &\quad + \alpha^2 e^{\alpha x}\cos(\beta x) + \beta^2 e^{\alpha x}\cos(\beta x) \\ &= (\alpha^2 - \beta^2 - 2\alpha^2 + \alpha^2 + \beta^2)\cos(\beta x) \\ &\quad + (-2\alpha\beta + 2\alpha\beta)\sin(\beta x) \\ &= 0 \end{aligned}$$

So, $f_1(x) = e^{\alpha x}\cos(\beta x)$ solves the ODE.

Similarly you can check that $f_2(x) = e^{\alpha x}\sin(\beta x)$ solves the ODE. Let's make sure these solutions are linearly independent.

We have

$$W\left(e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)\right)$$

$$= \begin{vmatrix} e^{\alpha x}\cos(\beta x) & e^{\alpha x}\sin(\beta x) \\ \alpha e^{\alpha x}\cos(\beta x) - \beta e^{\alpha x}\sin(\beta x) & \alpha e^{\alpha x}\sin(\beta x) + \beta e^{\alpha x}\cos(\beta x) \end{vmatrix}$$

$$= \alpha e^{2\alpha x}\cos(\beta x)\sin(\beta x) + \beta e^{2\alpha x}\cos^2(\beta x)$$
$$- \alpha e^{\alpha x}\sin(\beta x)\cos(\beta x) + \beta e^{2\alpha x}\sin^2(\beta x)$$

$$= \beta e^{2\alpha x}\left(\cos^2(\beta x) + \sin^2(\beta x)\right)$$

$= \beta e^{2\alpha x} \neq 0$ for any x since $\beta \neq 0$.

Conclusion: Every solution to the ODE is of the form

$$y_h = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$